ON SHOCK WAVE PROPAGATION IN AN ELASTOPLASTIC MEDIUM WITH HARDENING

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The problem of shock wave propagation in an elastoplastic medium with translational hardening is investigated [1].

A closed system of equations in jumps [2] has not been obtained successfully for many rheological materials. It is shown in [3] that in such cases a system of equations in discontinuities can be closed by using the analysis of the shock layer structure. This results in the need for a joint solution of the problems of shock wave propagation and structure.

The change in the discontinuous qualities within the transition layer in the problem of wave structure is described by a system of ordinary differential equations. Conditions for the existence and uniqueness of transient solutions of such systems have been studied sufficiently (*) and are assumed satisfied.

An exact solution of the combined problems of propagation and structure is difficult, hence as a simplification it is proposed to consider the linear dependence between "discontinuous" functions within the transition layer. On the basis of this method, a closed system of equations in jumps is successfully obtained below for an elastoplastic medium and the shock wave properties are investigated.

1. The material is assumed plastically incompressible, and its rheological model is pictured in Fig. 1. The dependence between the stresses s_{ij}^1, s_{ij}^2 and strains e_{ij}^1, e_{ij}^2 in the first and second elasticity elements is written, respectively, in the form of Hooke's law (1.1)

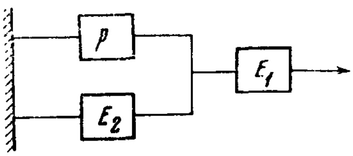

Fig. 1.

$$s_{ij}^1 = \lambda_1 e_{kk}^1 \delta_{ij} + 2\mu_1 e_{ij}^1, \qquad s_{ij}^{*2} = 2\mu_2 e_{ij}^{*2}$$

Here λ_1, μ_1 and μ_2 are elastic constants, and the asterisk superscript denotes the deviator part of the tensor. The stresses s_{ij}^p and the strain rate ε_{ji}^p in the plasticity element are connected by the associated flow law [1]

$$\varepsilon_{ij}^{p*} = \frac{\partial \varphi}{\partial t} s_{ij}^{p*}, \qquad \varepsilon_{kk}^p = 0 \tag{1.2}$$

The stresses s_{ij}^{p*} satisfy the Mises plasticity condition

$$s_{ij}^{p*} s_{ij}^{p*} = 2k^2 \qquad (k \text{ is the yield point}) \tag{1.3}$$

Since the elements P and E_2 are connected in parallel, the strains in these elements

(*) G. Ia. Liubarskii, Dissertation, Moscow, 1965.

coincide

$$e_{ij}^p = e_{ij}^2 \tag{1.4}$$

The strain e_{ij} and strain rate of the medium ε_{ij} are composed of strains and strain rates in the elements P and E_1, i.e.,

$$e_{ij} = e_{ij}^p + e_{ij}^1, \qquad \varepsilon_{ij} = \varepsilon_{ij}^p + \varepsilon_{ij}^1 \tag{1.5}$$

The stress in the medium σ_{ij} coincides with the stress in the element E_1 which equals the sum of the stresses in the elements P and E_2

$$\sigma_{ij} = s_{ij}^1 = s_{ij}^2 + s_{ij}^p \tag{1.6}$$

The joint solution of the system of equations (1.1), (1.2), (1.4) - (1.6) in σ_{ij}, e_{ij} and ε_{ij} results in the equations

$$\frac{\partial \sigma_{ij}^*}{\partial t} = 4(\mu_1\mu_2 e_{ij}^* - \mu_0\sigma_{ij}^*)\frac{\partial \varphi}{\partial t} + 2\mu_1\varepsilon_{ij}^* \tag{1.7}$$

$$\sigma_{kk} = (3\lambda_1 + 2\mu_1)e_{kk}, \qquad 2\mu_0 = \mu_1 + \mu_2$$

The plasticity condition results in the expression

$$\left(\sigma_{ij}^* - \frac{\mu_1\mu_2}{\mu_0}e_{ij}^*\right)\left(\sigma_{ij}^* - \frac{\mu_1\mu_2}{\mu_0}e_{ij}^*\right) = \frac{1}{2}\left(\frac{k\mu_1}{\mu_0}\right)^2 \tag{1.8}$$

The particle displacements u_i, velocity v_i, small strain tensor e_{ij} and strain rate tensor ε_{ij} are connected by the relations

$$v_i = \partial u_i / \partial t, \qquad e_{ij} = {}^1/_2(u_{i,j} + u_{j,i}) \tag{1.9}$$

$$\varepsilon_{ij} = \partial e_{ij} / \partial t = {}^1/_2(v_{i,j} + v_{j,i})$$

Let a surface of strong discontinuity Σ be propagated with velocity G in a continuum. The structure of the shock wave transition layer can be described by introducing additional viscous properties of the continuum which do not appear outside the shock layer. We then obtain the governing equations within the shock layer from (1.7) and (1.8) by replacing σ_{ij} by the difference $\sigma_{ij} - d_{ij}$, where d_{ij} is the stress tensor due to the additional viscous properties. However, at the level of the linear approximation of the dependence between the discontinuous shock quantities, the stresses d_{ij} introduce no contribution into the equation for the discontinuities [3]. Hence, within the transition layer the governing equations (1.7) and (1.8) will be utilized.

If f and ψ are two discontinuous functions, then the linear dependence

$$f(\psi) = f^+ + \frac{\psi - \psi^+}{[\psi]}[f], \qquad [f] = f^+ - f^- \tag{1.10}$$

between them will be utilized within the transition layer. The plus or minus superscript means that the value of the discontinuous quantity is taken ahead of or behind the shock front, respectively.

The dependence (1.10) permits writing (1.7) in jumps. To do this, let us integrate it across the transition layer from the rear to the forward shock front

$$\int_{x_n^-}^{x_n^+}\left(\frac{\delta\sigma_{ij}^*}{\delta t} - G\frac{\partial\sigma_{ij}^*}{\partial x_n}\right)dx_n - 2\mu_1\int_{x_n^-}^{x_n^+}\left(\frac{\delta e_{ij}^*}{\delta t} - G\frac{\partial e_{ij}^*}{\partial x_n}\right)dx_n =$$

$$= 4 \int_{x_n^-}^{x_n^+} \left(\frac{\delta\varphi}{\delta t} - G\frac{\partial\varphi}{\partial x_n}\right)(\mu_1\mu_2 e_{ij}{}^* - \mu_0\sigma_{ij}{}^*)\, dx_n \tag{1.11}$$

Here x_n is the normal coordinate to the transition layer, $x_n{}^+$ and $x_n{}^-$ are coordinates of the forward and rear shock fronts, respectively. Taking account of the properties of the delta-time-derivative, the left side of (1.11) is integrated to higher order accuracy. The right side can also be integrated if (1.10) is utilized. After the mentioned manipulations, we obtain the desired relationship in discontinuities

$$[\sigma_{ij}{}^*] - 2\mu_1 [e_{ij}{}^*] = 2[\varphi]\{2\mu_1\mu_2 e^{*+}_{ij} - 2\mu_0\sigma^{*+}_{ij} - \mu_1\mu_2[e_{ij}{}^*] + \\ + \mu_0[\sigma_{ij}{}^*]\} + \ldots \tag{1.12}$$

The three dots in (1.12) denote small terms containing jumps of higher degree than the second.

From the second equation in (1.7) and the plasticity condition (1.8) we find

$$[\sigma_{kk}] = (3\lambda_1 + 2\mu_1)[e_{kk}] \tag{1.13}$$

$$\left[\sigma_{ij}{}^* - \frac{\mu_1\mu_2}{\mu_0} e_{ij}{}^*\right]\left\{2\left(\sigma_{ij}{}^* - \frac{\mu_1\mu_2}{\mu_0} e_{ij}{}^*\right)^+ - \left[\sigma_{ij}{}^* - \frac{\mu_1\mu_2}{\mu_0} e_{ij}{}^*\right]\right\} = 0 \tag{1.14}$$

The kinematic and dynamic conditions of compatibility of the discontinuities

$$[\sigma_{ij}]\nu_j = -\rho^+ G[v_i], \quad [u_{i,j}] = \omega_i\nu_j, \quad [v_i] = -G\omega_i \tag{1.15}$$

close the system of equations (1.12) - (1.14) where ν_i is the normal to the surface of discontinuities.

The auxiliary relationships

$$]\sigma_{ij}{}^*]\nu_i\nu_j = (\rho^+G^2 - \lambda_1 - {}^2/_3\,\mu_1)\,\omega_n \tag{1.16}$$
$$[e_{ij}{}^*]\nu_i\nu_j = {}^2/_3\,\omega_n, \qquad \omega_n = \omega_i\nu_i$$

can be obtained from (1.15). Multiplying (1.12) by $\nu_i\nu_j$ and utilizing (1.16), we obtain

$$2[\varphi] = F_1\omega_n(\mu_0\omega_n F_2 - \mu_1 s^{p}_{nn}{}^+)^{-1} \tag{1.17}$$

$$F_1 = \rho^+G^2 - \lambda_1 - 2\mu_1, \; F_2 = \rho^+G^2 - \lambda_1 - {}^2/_3\,\mu_1(1 + \mu_2/\mu_0)$$

where the subscript n denotes the normal direction. Here and henceforth, there will be no summation over this repeated subscript.

The quantity φ characterizes the irreversible plastic deformations of a continuum. If the jump in this quantity on the shock is zero, then the plastic deformations are continuous, and such a surface of strong discontinuity is called neutral. It is seen from (1.17) that $[\varphi] = 0$, if $\rho^+G^2 = \lambda_1 + 2\mu_1$ or when $\omega_n = 0$. If $\omega_n = 0$ and $[\varphi] = 0$, then it follows from (1.12) and (1.15) that $\rho^+G^2 = \mu_1$, i.e., neutral shocks in the elastoplastic medium with hardening, just as in an elastoplastic medium [3], can be propagated only with two discontinuous velocities.

It is seen from (1.17) that in the general case $[\varphi]$ depends on ω_n. In the particular case $s^{p\,k+}_{nn} = 0$ the quantity $[\varphi]$ is independent of ω_n and is expressed in terms of the shock velocity. This case will be investigated below.

Solving (1.12) for $[\sigma_{ij}^*]$ we obtain

$$[\sigma_{ij}^*] = \{\mu_1 (1 - \mu_2 [\varphi]) (\omega_i \nu_j + \omega_j \nu_i - {}^2/_3\, \omega_n \delta_{ij}) - \\ - 2 [\varphi] \mu_1 s_{ij}^{p*+}\} (1 - 2\mu_0 [\varphi])^{-1} + \ldots \qquad (1.18)$$

Substituting (1.13) and (1.18) into (1.15) results in the equation

$$\{\rho^+ G^2 \omega_i - (\lambda_1 + {}^2/_3\, \mu_1) \omega_n \nu_i\} (1 - 2\mu_0 [\varphi]) = \\ = \mu_1 (1 - \mu_2 [\varphi]) (\omega_i + {}^1/_3\, \omega_n \nu_i) - 2 s_{ni}^{p*+} \mu_1 [\varphi] + \cdots \qquad (1.19)$$

Let us eliminate $[\varphi]$ from (1.19) by using (1.17)

$$F_1 \{[2\lambda_1 \mu_0 + {}^1/_3\, \mu_1 (\mu_2 + 4\mu_0)] \omega_n^2 \nu_i + (\mu_1 \mu_2 - 2\mu_0 \rho^+ G^2) \omega_n \omega_i\} + 2\mu_1 s_{ni}^{p*+} F_1 \omega_n = \\ = 2 (\mu_0 \omega_n F_2 - \mu_1 s_n^{p*+}) \{(\lambda_1 + \mu_1) \omega_n \nu_i - F_3 \omega_i\} + \ldots \qquad (1.20)$$

$$F_3 = \rho^+ G^2 - \mu_1$$

Only two among the three euqations (1.20) are independent. To simplify the subsequent calculations, let us introduce a local coordinate system with origin moving together with the surface of discontinuity Σ; we direct the x_3-axis along the normal to Σ. Setting $i = \alpha$ $(\alpha = 1, 2)$, we obtain

$$F_1 (\mu_1 \mu_2 - 2\mu_0 \rho^+ G^2) \omega_3 \omega_\alpha + F_1 \mu_1 \omega_3 s_{3\alpha}^{p+} + 2F_3 (\mu_0 \omega_3 F_2 - \mu_1 s_{33}^{p**}) \omega_\alpha = 0 \quad (1.21)$$

Hence ω_α is expressed in terms of ω_3. Elimination of $[\varphi]$, ω_α and $[\sigma_{ij}^*]$ from (1.14) by using (1.21), (1.17) and (1.18) results in a fourth power equation in $\rho^+ G^2$ Omitting the intermediate calculations, we write the final form of this equation

$$AF_3 \{3\mu_1^2 F_1 s_{\alpha 3}^{p+} s_{\alpha 3}^{p+} s_{33}^{p*+} + 3\mu_1^2 F_3 (s_{33}^{p*+})^3 + 6k^2 \mu_0 F_3 F_1 s_{33}^{p*+} - \mu_1^3 \omega_3 F_4 (s_{33}^{p*+})^2 - \\ - 6\mu_0 \mu_1 \omega_3 F_1 F_2 s_{\alpha 3}^{p+} s_{\alpha 3}^{p+} - 6\mu_0 \mu_1 \omega_3 F_2 F_3 (s_{33}^{p*+})^2 - k^2 \mu_0 \mu_1 \omega_3 F_1 F_4\} = 0 \quad (1.22)$$

$$A = \mu_0 F_2 \omega_3 - \mu_1 s_{33}^{p*+}, \qquad F_4 = \rho + G^2 + 3\lambda_1 + 2\mu_1$$

The first root of this equation $\rho^+ G^2 = \mu_1$ corresponds to a transverse neutral wave on which there are no plastic deformation discontinuities and $[\varphi] = 0$. Such waves will be investigated in more detail below. Equating the factor A to zero, we find another root for $\rho^+ G^2$. But such a shock cannot be realized since otherwise $[\varphi] \to \infty$, which is impossible. To find the two remaining roots of (1.22) let us equate the expression in the braces to zero, and we obtain a quadratic equation. Therefore, two plastic shocks are possible in a three-dimensional elastoplastic medium. In the case of very low intensity shocks, the mentioned surfaces of discontinuities are propagated with the velocities

$$\rho^+ G_{1,2}^2 = \frac{1}{2}\lambda_1 + \frac{3}{2}\mu_1 - \frac{\mu_1^2}{4\mu_0 k^2} s_{i3}^{p*+} s_{i3}^{p*+} \pm \frac{1}{2} \left\{ \left[\lambda_1 + \mu_1 + \frac{\mu_1^2}{2\mu_0 k^2} (s_{\alpha 3}^{p+} s_{\alpha 3}^{p+} - (s_{33}^{p*+})^2) \right]^2 + \\ + \frac{\mu_1^4}{\mu_0^2 k^4} s_{\alpha 3}^{p+} s_{\alpha 3}^{p+} (s_{33}^{p*+})^2 \right\}^{1/2} \qquad (1.23)$$

2. Let us consider particular cases.

1. Neutral Waves. The plastic deformation components are continuous on neutral shocks, which is expressed by the condition $[\varphi] = 0$. We then obtain two possible cases from (1.17)

$$\omega_3 = 0, \ \rho^+ G^2 = \lambda_1 + 2\mu_1$$

For these cases, we have from (1.18) and (1.15) respectively

$$\rho^+ G_*^2 = \mu_1, \qquad \omega_3 \neq 0; \qquad \omega_3 = 0, \ \omega_\alpha = 0$$

Therefore, neutral waves have the properties of elastic waves.

The state of stress in front of and behind the shock should satisfy the plasticity condition (1.14), which reduces, by using (1.18), to

$$(s_{i3}^{p*+} + s_{i3}^{p*-})\,\omega_i = 0 \tag{2.1}$$

This condition means that the mean stress vector on the plasticity element in the area element of the surface of discontinuities is orthogonal to the vector of the velocity discontinuity. For very weak shocks, when squares of the jumps ω_i can be neglected, condition (2.1) simplifies to

$$s_{i3}^{p*+}\omega_i = 0 \tag{2.2}$$

The converse can also be proved: If condition (2.1) is satisfied, then the shock is neutral. For the proof it is sufficient to substitute (1.18) into (1.14) and to utilize condition (2.5). Afterwards, we obtain $[\varphi] = 0$.

Let us examine the two simplest loadings of a rectangular plate $ABCD$ (Figs. 2a, b). The plate material is elastoplastic. Then condition (2.2) becomes

$$\sigma_{i3}^{*+}\omega_i = 0 \tag{2.3}$$

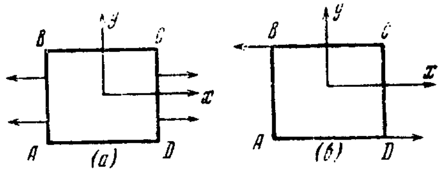

Fig. 2.

In case (a) the plate is stretched by certain stress resultants applied to the faces AB and CD. In case (b) shear stresses are applied to the faces BC and AD of the plate. In both cases, both the elastic and plastic parts of the strain are identical at each point of the plate. Let us suddenly apply a small loading to the face AD along the normal, or to the face AB along it. In case (a) condition (2.3) will be satisfied for both experiments, and hence, a neutral shock, longitudinal in the first case and transverse in the second, will be propagated along the plate. In case (b) longitudinal neutral waves originate if a normal impact is made on the faces AB or AD.

2. Spherical symmetry of the state of stress on the surface of discontinuity. In this case the following equalities are valid

$$s_{11}^{p*} = s_{22}^{p*} = -\frac{1}{2} s_{33}^{p*} = \pm\frac{\sqrt{3}}{3}k, \quad s_{12}^{p} = s_{13}^{p} = s_{23}^{p} = 0, \quad \omega_\alpha = 0 \tag{2.4}$$

The last equality from (2.4) indicates that the shock is irrotational. It results from (1.23) that one weak plastic wave degenerates into a transverse neutral wave, and the other is propagated at the velocity

$$\rho^+ G^2 = \lambda_1 + 2\mu_1 - {}^2/_3\,\mu_1^2\mu_0^{-1} \tag{2.5}$$

From the associated flow law (1.2) and (1.18) we obtain

$$[e_{11}^p] = \frac{s_{11}^{p*}\omega_3}{\mu_2\omega_3 - 3s_{11}^{p*}} = [e_{22}^p] = -\frac{1}{2}[e_{33}^p] \approx -\frac{1}{3}\omega_3$$

$$[e_{12}^p] = [e_{13}^p] = [e_{23}^p] = 0$$

3. The Case $s_3^{p*+} = 0$. From (1.22) we find that one plastic wave degenerates into a nautral longitudinal wave, and the other is propagated at the velocity

$$\rho^+ G^2 (s_{\alpha 3}^{p+} s_{\alpha 3}^{p+} + {}^1/_6 k^2) = (\lambda_1 + {}^2/_3\mu_1)(s_{\alpha 3}^{p+} s_{\alpha 3}^{p+} - {}^1/_2 k^2) + {}^2/_3\, \mu_1\mu_2\mu_0^{-1} s_{\alpha 3}^{p+} s_{\alpha 3}^{p+} \tag{2.6}$$

The quantity $s_{\alpha 3}^{p+} s_{\alpha 3}^{p+}$ can vary between the limits 0 and k^2. For a certain value of this quantity the shock velocity vanishes, and for lesser values the right side of (2.6) will be negative, and this surface of strong discontinuities becomes impossible. The velocity of the acoustic wave for this case is found from (1.23) as

$$\rho^+ G^2 = \mu_1 - {}^1/_2 \mu_1^2 \mu_0^{-1} k^{-2} s_{\alpha 3}^{p+} s_{\alpha 3}^{p+} \tag{2.7}$$

From a comparison of (2.6) and (2.7) it results that the shock (2.6) does not go over into the corresponding acoustic wave when the shock intensity goes to zero.

If $s_{\alpha 3}^{p+} s_{\alpha 3}^{p+} = k^2$, which is possible only under the conditions $s_{11}^{p*+} = s_{22}^{p*+} = s_{33}^{p*+} = 0$, then we find $\rho^+ G^2 = \mu_1\mu_2(\mu_1 + \mu_2)^{-1}$ from (2.7) for the velocity of propagation of the equivolume wave. The existence of this wave is specified by the presence of the elastic properties and the hardening properties simultaneously. If the material does not posses at least one of the mentioned properties, then such a wave cannot be propagated.

The second law of thermodynamics, according to which energy dissipation because of plastic flow is nonnegative

$$\varepsilon_{ij}^{p*} s_{ij}^{p*} = 2k^2 \partial\varphi / \partial t \geqslant 0 \tag{2.8}$$

imposes a constraint on shock propagation.

Hence, a constraint on the discontinuity in the normal velocity component can be obtained. To do this, we integrate (2.8) across the transition layer from $-h$ to $+h$. Since the integration is performed as the argument increases, the sign of the inequality is conserved. We then obtain $[\varphi] \leqslant 0$ from (2.8). Substituting this inequality into (1.17) considering ω_n small, we find $\omega_n s_{nn}^{p*+} \leqslant 0$. This inequality indicates that only one rarefaction shock is possible in the domain of material tension, and only a compression wave in the compression domain. It should be noted that the equality sign corresponds here to a neutral shock.

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